Negative Translations and Normal Modality

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Abstract

We discuss the behaviour of variants of standard negative translations— Kolmogorov, Gödel-Gentzen, Kuroda and Glivenko—in extensions of the propositional calculus with an unary normal modality and additional axioms. More specifically, we address the question whether negative translations as a rule embed faithfully a classical modal logic into its intuitionistic counterpart. As it turns out, even the Kolmogorov translation can go wrong with rather natural modal principles. Nevertheless, we isolate sufficient syntactic criteria ensuring adequacy of well-behaved (or, in our terminology, *regular*) translations for logics axiomatized with formulas satisfying these criteria, which we call *enveloped implications*. Furthermore, a large class of computationally relevant modal logics—namely, logics of type inhabitation for *applicative functors* (a.k.a. idioms)—turns out to validate the modal counterpart of the Double Negation Shift, thus ensuring adequacy of even the Glivenko translation. All our positive results are proved purely syntactically, using the minimal natural deduction system of Bellin, de Paiva and Ritter extended with additional axioms/combinators, hence also allowing a direct formalization in a proof assistant (in our case Coq).

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1 Introduction

Consider the simplest possible modal extension $\mathcal{L}_\Box$ of the signature of IPC with a single normal $\Box$. Take the standard notion of a *logic* $\mathrm{i}_\Box\mathcal{Z}$ in such a language [12, 26, 50, 56, 58, 59] (§2) and whichever variant t of standard negative translations the reader prefers (§3). Is such a translation t by default *adequate* for every $\mathrm{i}_\Box\mathcal{Z}$—that is, given any $\phi \in \mathcal{L}_\Box$, is it the case that $\phi \in \mathrm{c}_\Box\mathcal{Z} = \mathrm{i}_\Box\mathcal{Z} + \mathsf{CPC}$ iff $\phi^t \in \mathrm{i}_\Box\mathcal{Z}$ (§4)? If not, is it possible to find some general criteria on the axiomatization of $\mathrm{i}_\Box\mathcal{Z}$ which ensure adequacy—and show correctness of these criteria in a purely syntactic way (§5)? Can it happen that for entire classes of logic with good computational, type-theoretic and categorical motivation the simplest possible translation (Glivenko) would be adequate (§6)?

To the best of our knowledge, while the question of adequacy of negative translations (we call this property *$\neg\neg$-incompleteness*) has been addressed for the minimal system $\mathrm{i}_\Box\mathsf{K}$ [12, 28] (see also §7 for a discussion of [8]), there has been no systematic study for arbitrary axioms. We found the shortage of relevant research puzzling for several reasons, even disregarding Curry-Howard interpretations of intuitionistic modalities (cf., e.g., [6, 31, 36, 44, 45] for overview and more references) and negative translations (cf., e.g., [1, 27, 39] and [49, Ch. 6]).

In the light of the *Standard Translation* (see, e.g., [59, p. 135]), a normal modal $\Box$ can be thought of as a restricted form of $\forall$—yielding a calculus/theory intermediate between the intuitionistic propositional calculus (IPC) and the intuitionistic predicate calculus (IQC), for which the behaviour of negative translations is well-understood. Negative translations

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(mostly Glivenko) have been also studied for other non-classical logic, especially substructural ones [20, 23, 43]. The Gödel-Gentzen translation also plays an important rôle in the recently developed theory of *possibility semantics* for modal logic [7, 28]. Moreover, our lack of systematic knowledge regarding $\neg\neg$-completeness contrasts with the existence of exhaustive studies of Gödel-McKinsey-Tarski-type translations of $\mathcal{L}_\Box$-logics [12, 56, 58].

On the other hand, negative translations can quickly go very wrong with natural and important extensions of intuitionistic logic. As we are going to discuss in §4.1, a striking example is provided by the logic of bunched implications BI. Thus, beginning the systematic study of modal negative translations with a single normal $\Box$ seems the right level of generality at the moment, especially that such a (dual) operator can be seen not only as a restricted universal quantifier, but also as the object-level trace of an endofunctor (lax monoidal wrt the cartesian structure) or the type-inhabitation-level trace of a type constructor distributing over conjunctions/products.

Besides, a systematic investigation of $\neg\neg$-completeness seems warranted by Simpson's [47, Ch 3.2] influential *requirements* that an intuitionistic modal logic is supposed to satisfy, in particular, Requirement 3: *The addition of $\alpha \vee \neg\alpha$ to an intuitionistic modal logic should yield a standard classical modal logic.* On the one hand, the requirement is too restrictive: there are entire classes of important intuitionistic modal logics which classically collapse to rather trivial systems. See the discussion of $\mathrm{i}_\Box\mathsf{R}$ and its extensions like PLL and SL in §6, esp. Remark 25 (ironically, even the Glivenko translation is adequate for such logics); other "classically near-degenerate" examples can be found in provability logic of extensions of HA, cf., e.g., [29, 54]. On the other hand, there is something wrong with the relationship between $\mathrm{i}_\Box\mathcal{Z}$ and $\mathrm{c}_\Box\mathcal{Z}$, if negative translations are not adequate—and as it turns out, this can happen with very simple *modal reduction principles* [57, §4.5]. Contrast for example $\mathsf{4}$, i.e., the transitivity axiom $\Box p \to \Box\Box p$, with axioms like $\mathsf{C4}$, i.e., $\Box\Box p \to \Box p$ or $\mathsf{Minus4}$, i.e., $\Box\Box p \to \Box\Box\Box p$. In the light of our Enveloped Implication Theorem, i.e., Theorem 18, the logic $\mathrm{i}_\Box\mathsf{4}$ is $\neg\neg$-complete (Corollary 19), but $\mathrm{i}_\Box\mathsf{C4}$ (Example 11 and Theorem 12) or $\mathrm{i}_\Box\mathsf{Minus4}$ (Example 13 and Theorem 14) are not.

An interesting feature of general positive results in the present paper is that we were able to show them purely syntactically. This contrasts with less constructive methods typically employed in *modal logic as "die Klassentheorie"* [57], cf. §2 for a discussion of these points and more details. Note that a similar approach was pursued in investigation of the Glivenko translation for substructural logics by Ono and coauthors [20, 43]. This allows a straightforward formalization in a proof assistant (plain Coq in our case, but translating this development to any other setting would be straightforward: we only used a small number of self-designed tacticals to shorten proof scripts), to which the reader is referred for omitted details of proofs.[1] It is possible, however, to provide semantic characterizations of $\neg\neg$-completeness: this is briefly discussed in §7 and the details are postponed to either the journal version or to a companion paper.

[1] To download, `git clone git://git8.cs.fau.de/dnegmod`. Web fronted available at `https://cal8.cs.fau.de/redmine/projects/dnegmod`. As described in the README file, full documentation can be produced by `make all-gal.pdf`. We have tested the development in several versions of Coq, ranging from Coq 8.4pl4 (November 2015) to Coq 8.6 (April 2017).

2 Logics and proof systems

Modal formulas over a supply of propositional variables Σ (unless stated otherwise, fixed and dropped from the notation) are defined by

$$\mathcal{L}_\Box\Sigma \quad \phi, \psi ::= \bot \mid p \mid \phi \to \psi \mid \phi \wedge \psi \mid \phi \vee \psi \mid \Box\phi$$

where $p \in \Sigma$. $\mathcal{L}$ denotes the pure propositional language (i.e., without $\Box$). We follow standard conventions regarding binding priorities and associativity of connectives; in particular, we assume $\to$ associates to the right. As usual, $\neg\phi$ is $\phi \to \bot$. To save space and improve readability, let us write $\neg\neg\phi$ as $\neg\!\neg\phi$.[2] Also, as usual, a substitution is a function $s : \Sigma \mapsto \mathcal{L}_\Box$. Its uniquely determined extension $\overline{s} : \mathcal{L}_\Box \mapsto \mathcal{L}_\Box$ will be notationally conflated with s itself. A particularly important substitution $s_{\neg\!\neg}(\cdot)$ is generated by $s_{\neg\!\neg}(p) := \neg\!\neg p$ for each $p \in \Sigma$.

Unlike the classical case, it matters that $\Box$ is primitive and $\Diamond$ is not. There are several possible approaches to combining $\Box$ and $\Diamond$ in the intuitionistic setting (overviews can be found, e.g., in [12, 47, 56, 58]); to save space and clarity, we leave the discussion of negative translations in each of these setups for the full version.

2.1 Intuitionistic modal logics á la Hilbert

As usual, by a(*n intuitionistic normal modal*) *logic* we understand a set of formulas closed under the axioms of intuitionistic propositional calculus (IPC),

K $\quad \Box(p \to q) \to \Box p \to \Box q$,

Modus Ponens: $\dfrac{\phi \to \psi \qquad \phi}{\psi}$, the rule of *Necessitation*, also known as the *Gödel rule*: $\dfrac{\phi}{\Box\phi}$, and, finally, *substitution*: $\dfrac{\phi}{s(\phi)}$ for any s.

The smallest set of formulas closed under these rules is denoted as $\mathrm{i}_\Box\mathsf{K}$. For any $\mathcal{Z} \subseteq \mathcal{L}_\Box$, the smallest intuitionistic modal logic containing $\mathcal{Z}$—i.e., the normal extension of $\mathrm{i}_\Box\mathsf{K}$ axiomatized by $\mathcal{Z}$—is denoted by $\mathrm{i}_\Box\mathcal{Z}$.[3] Furthermore, $\mathrm{c}_\Box\mathcal{Z}$ is the smallest logic containing $\mathcal{Z}$ and the classical propositional calculus (CPC); we can formally define it as $\mathrm{i}_\Box\mathcal{Z} + \neg\!\neg p \to p$. Note here that for sets of axioms, it is notationally convenient to omit singleton brackets and replace $\cup$ by $+$. The smallest modal logic containing CPC is denoted as $\mathrm{c}_\Box\mathsf{K}$. Finally, let us write $\phi \vdash_{\mathrm{i}_\Box\mathcal{Z}} \psi$ for $\phi \to \psi \in \mathrm{i}_\Box\mathcal{Z}$ and $\phi \dashv\vdash_{\mathrm{i}_\Box\mathcal{Z}} \psi$ for $\phi \leftrightarrow \psi \in \mathrm{i}_\Box\mathcal{Z}$.

Such a Hilbert-style definition of a logic is rather inconvenient for proof search or, indeed, for most proofs by structural induction on formulas. Nevertheless, it lends itself to a natural process of *algebraization*, Lindenbaum-Tarski style: in the case of intuitionistic (or classical) $\Box$ one obtains expansions of Heyting (or Boolean) algebras with equational axioms capturing that $\Box$ distributes over arbitrary finite (possibly empty) conjunctions. Additional equations encode additional axioms from $\mathcal{Z}$. Futhermore, such algebras can be dually represented as *Esakia spaces*, *Priestley spaces*, *descriptive general frames* or (in the classical case) *Stone coalgebras for the Vietoris functor* [32], which allows topological or model-theoretic technology in investigating large classes of logics. The use of algebras and/or their duals and the need to formalize a reasonably large fragment of their metatheory, however, complicates formalization in proof assistants, limiting their transparency, transferability and the potential for program

[2] This notation indicates the perspective on double negation as a modality, especially over negation-free and bottom-free intutionistic sytax [11, 17]. We will discuss this briefly in §7.

[3] Conceivable notational conventions to be found in the literature include also, e.g., $\mathsf{iK}_\Box + \mathcal{Z}$ or $\mathsf{IntK}_\Box \oplus \mathcal{Z}$.

extraction and computational interpretation. It is also worth adding that most standard approaches to this line of investigation are rather non-constructive; even in purely algebraic research, one often relies on various consequences of the Axiom of Choice (such as the existence of a subdirect decomposition of any algebra) and still more so in representation and duality results leading to classes of frames, spaces and coalgebras mentioned above.[4]

2.2 Intuitionistic modal logics á la Gentzen

For logics with additional axioms of a specific syntactic shape, there are generic methods producing, e.g., suitable hyper-sequent calculi allowing cut-elimination. Especially in the substructural setting, the scope of such methods is systematically studied by *algebraic* or *systematic proof theory* (cf., e.g., [15, 16]), but this line of work also reveals that there are fundamental limitations on the shape of suitable axioms. Similar restrictions apply to labelled natural-deduction calculi for extensions of $\mathrm{i}_\Box\mathsf{K}$ (cf., e.g., [47, §6.3]). And, needless to say, there are many examples of ostentatiously misbehaving formulas, including those making resulting logics undecidable.

Is there any chance then of using Gentzen-inspired methods to prove general characterization results for *arbitrary* axioms such as our Theorems 15 and 18 below?

The solution turns out to be surprisingly simple: we do not need to postulate unrestricted substitution as an explicit rule. As first observed, to the best of our knowledge, by Sobociński (in a Hilbert-style setting) [34, 48], propositional deductions can be put in a form where the substitution rule is *only applied to axioms*. This idea can be replayed in a Gentzen-style setup. Obviously, we cannot assume that such systems with additional axioms would allow in general results like normalization/cut-elimination, not to mention Martin-Löf-style *local soundness* and *local completeness* [45]. But they will do a perfectly fine job in improving the support for structural induction on formulas, and this is all we need in this paper.

Given the simplicity of our goals, there is no need to complicate our presentation (and the associated Coq formalization) with additional apparatus like labels, multiple contexts, nested sequents, hypersequents etc. Instead, just like in the work of Kakutani [30], we take as our base the standard, single-context ND-calculus for $\mathrm{i}_\Box\mathsf{K}$ by Bellin, de Paiva and Ritter [5, 6, 44]—with additional tweaks such as not only presenting everything in a sequent notation, but explicitly using multisets. Essentially, this yields a calculus equivalent to those obeying the so-called *Complete Discharge Convention* [51],[53, §2.1.10].

Thus, let $\Gamma, \Gamma', \Delta \ldots$ range over finite multisets of formulas from $\mathcal{L}_\Box$. As usual, let $\bigwedge \Gamma$ be the conjunction of all formulas from Γ. For any given set of axioms $\mathcal{Z}$, the rules of its corresponding calculus $\mathsf{Ni}_\Box\mathcal{Z}$ governing derivability of judgements of the form $\Gamma \Rightarrow \phi$ are given in Figure 1. In proofs, one can freely use rules derivable in in $\mathsf{Ni}_\Box\mathsf{K}$, such as

$$\neg\neg\!\to_1 \frac{\vdash_{\mathsf{Ni}_\Box\mathcal{Z}} \Rightarrow \phi \to \psi}{\vdash_{\mathsf{Ni}_\Box\mathcal{Z}} \Rightarrow \neg\neg\phi \to \neg\neg\psi} \qquad \neg\neg\!\to_2 \frac{\vdash_{\mathsf{Ni}_\Box\mathcal{Z}} \Rightarrow \phi \to \psi}{\vdash_{\mathsf{Ni}_\Box\mathcal{Z}} \Rightarrow \neg\neg(\neg\neg\phi \to \neg\neg\psi)}$$

▶ **Theorem 1.** *For any $\mathcal{Z}$, Γ and α, $\vdash_{\mathsf{Ni}_\Box\mathcal{Z}} \Gamma \Rightarrow \alpha$ iff $\bigwedge \Gamma \to \alpha \in \mathrm{i}_\Box\mathcal{Z}$.*

2.3 Semantics

We keep discussion of semantics to a minimum. We only need intuitionistic Kripke frames to disprove validity of certain formulas. We refer the reader to numerous references [12, 26,

[4] There are more constructive approaches, but they are not often used by modal logicians studying entire lattices of logics; thus, their metatheory and the body of relevant results on offer are less developed.

Figure 1 Rules of $\mathsf{Ni}_{\Box}\mathcal{Z}$

Intuitionistic propositional rules:

$$\textsc{In}\ \frac{}{\vdash_{\mathsf{Ni}_{\Box}\mathcal{Z}} \phi, \Gamma \Rightarrow \phi} \qquad \top_I\ \frac{}{\vdash_{\mathsf{Ni}_{\Box}\mathcal{Z}} \Gamma \Rightarrow \top} \qquad \bot_E\ \frac{\vdash_{\mathsf{Ni}_{\Box}\mathcal{Z}} \Gamma \Rightarrow \bot}{\vdash_{\mathsf{Ni}_{\Box}\mathcal{Z}} \Gamma \Rightarrow \phi}$$

$$\rightarrow_I\ \frac{\vdash_{\mathsf{Ni}_{\Box}\mathcal{Z}} \Gamma, \phi \Rightarrow \psi}{\vdash_{\mathsf{Ni}_{\Box}\mathcal{Z}} \Gamma \Rightarrow \phi \rightarrow \psi} \qquad \rightarrow_E\ \frac{\vdash_{\mathsf{Ni}_{\Box}\mathcal{Z}} \Gamma \Rightarrow \phi \rightarrow \psi \qquad \vdash_{\mathsf{Ni}_{\Box}\mathcal{Z}} \Gamma \Rightarrow \phi}{\vdash_{\mathsf{Ni}_{\Box}\mathcal{Z}} \Gamma \Rightarrow \psi}$$

$$\wedge_I\ \frac{\vdash_{\mathsf{Ni}_{\Box}\mathcal{Z}} \Gamma \Rightarrow \phi \qquad \vdash_{\mathsf{Ni}_{\Box}\mathcal{Z}} \Gamma \Rightarrow \psi}{\vdash_{\mathsf{Ni}_{\Box}\mathcal{Z}} \Gamma \Rightarrow \phi \wedge \psi} \qquad \wedge_{E1}\ \frac{\vdash_{\mathsf{Ni}_{\Box}\mathcal{Z}} \Gamma \Rightarrow \phi \wedge \psi}{\vdash_{\mathsf{Ni}_{\Box}\mathcal{Z}} \Gamma \Rightarrow \phi} \qquad \wedge_{E2}\ \frac{\vdash_{\mathsf{Ni}_{\Box}\mathcal{Z}} \Gamma \Rightarrow \phi \wedge \psi}{\vdash_{\mathsf{Ni}_{\Box}\mathcal{Z}} \Gamma \Rightarrow \psi}$$

$$\vee_{I1}\ \frac{\vdash_{\mathsf{Ni}_{\Box}\mathcal{Z}} \Gamma \Rightarrow \phi}{\vdash_{\mathsf{Ni}_{\Box}\mathcal{Z}} \Gamma \Rightarrow \phi \vee \psi} \qquad \vee_{I2}\ \frac{\vdash_{\mathsf{Ni}_{\Box}\mathcal{Z}} \Gamma \Rightarrow \psi}{\vdash_{\mathsf{Ni}_{\Box}\mathcal{Z}} \Gamma \Rightarrow \phi \vee \psi}$$

$$\vee_E\ \frac{\vdash_{\mathsf{Ni}_{\Box}\mathcal{Z}} \Gamma \Rightarrow \phi \vee \psi \qquad \vdash_{\mathsf{Ni}_{\Box}\mathcal{Z}} \phi, \Gamma \Rightarrow \chi \qquad \vdash_{\mathsf{Ni}_{\Box}\mathcal{Z}} \psi, \Gamma \Rightarrow \chi}{\vdash_{\mathsf{Ni}_{\Box}\mathcal{Z}} \Gamma \Rightarrow \chi}$$

The Bellin, de Paiva and Ritter rule for $\mathsf{i}_{\Box}\mathsf{K}$ [5, 6, 30, 44]:

$$\Box_{\mathsf{K}}\ \frac{\vdash_{\mathsf{Ni}_{\Box}\mathcal{Z}} \Gamma \Rightarrow \Box\phi_1 \qquad \ldots \vdash_{\mathsf{Ni}_{\Box}\mathcal{Z}} \Gamma \Rightarrow \Box\phi_n \qquad \vdash_{\mathsf{Ni}_{\Box}\mathcal{Z}} \phi_1, \ldots, \phi_n \Rightarrow \psi}{\vdash_{\mathsf{Ni}_{\Box}\mathcal{Z}} \Gamma \Rightarrow \Box\psi}$$

The Sobociński-style rule for additional axioms [34, 48]:

$$\textsc{AxSb}\ \frac{\zeta \in \mathcal{Z} \qquad s \text{ a substitution}}{\vdash_{\mathsf{Ni}_{\Box}\mathcal{Z}} \Gamma \Rightarrow s(\zeta)}$$

36, 47] for a more detailed discussion. Thus, recall that a(n *intuitionistic* $\Box$-) *frame* is of the form $\mathfrak{F} := \langle W, \uparrow, \rightsquigarrow \rangle$, where $\uparrow \ \subseteq \ W \times W$ is a partial order on W and $\rightsquigarrow \ \subseteq \ W \times W$ satifies $\uparrow ; \rightsquigarrow ; \uparrow \ \subseteq \ \rightsquigarrow$, where ";" is the relational composition; in other words, $\rightsquigarrow$ is closed under pre- and postfixing with $\uparrow$. A valuation V maps elements of Σ to subsets of W, which are upward closed wrt $\uparrow$. It is used to define a forcing relation $\Vdash$ between elements of W and formulas of $\mathcal{L}_{\Box}$ using the usual clauses for intuitionistic connectives with $\uparrow$ used to interpret implication and with $\rightsquigarrow$ used to interpret $\Box$, i.e.,

$$\mathfrak{F}, V, w \Vdash \Box\phi \text{ if for any } w' \text{ s.t. } w \rightsquigarrow w', \text{ we have that } \mathfrak{F}, V, w' \Vdash \phi.$$

The interaction condition ensures that denotations of all formulas are upward closed wrt $\uparrow$; in fact, even a much weaker one would do, but as long as normal $\Box$ is the only additional modality, imposing $\uparrow ; \rightsquigarrow ; \uparrow \ \subseteq \ \rightsquigarrow$ is harmless [12, 26, 36, 47]. We write $\overline{V}$ for the inductive extension of V to all formulas, i.e., $\overline{V}(\phi)$ is the set of all points where ϕ holds. It is well-known (and trivial to check) that the set of all formulas which hold in entire W under every valuation is a logic. Finally, when representing countermodels graphically, we will use $\bullet$ for $\rightsquigarrow$-irreflexive points and $\circ$ for $\rightsquigarrow$-reflexive ones.

2.4 Intuitionistic double negation laws

While this paper is concerned with propositional logic, in some places it will be useful to contrast double negation laws in modal and in predicate logic. We write "IQC" for the intuitionistic (first-order) predicate calculus. To save space and avoid distractions, the reader is referred to e.g., Troelstra and van Dalen [52, Ch. 2] for an axiomatization.

Intuitionistic laws for $\neg\neg$ relevant from our point of view are summarized below. A fuller list can be found, e.g., in Ferreira and Oliva [21], where the reader is referred to for proofs:

▶ **Lemma 2.** *We have that:*

$$\neg\neg\phi \wedge \neg\neg\psi \dashv\vdash_{\mathsf{IPC}} \neg\neg(\neg\neg\phi \wedge \neg\neg\psi) \quad (1)$$
$$\neg\neg(\phi \wedge \psi) \dashv\vdash_{\mathsf{IPC}} \neg\neg(\neg\neg\phi \wedge \neg\neg\psi) \quad (2)$$
$$\neg\neg\phi \to \neg\neg\psi \dashv\vdash_{\mathsf{IPC}} \neg\neg(\neg\neg\phi \to \neg\neg\psi) \quad (3)$$
$$\neg\neg(\phi \vee \psi) \dashv\vdash_{\mathsf{IPC}} \neg\neg(\neg\neg\phi \vee \neg\neg\psi) \quad (4)$$
$$\neg(\neg\phi \wedge \neg\psi) \dashv\vdash_{\mathsf{IPC}} \neg\neg(\neg\neg\phi \vee \neg\neg\psi) \quad (5)$$
$$\neg\neg(\phi \to \psi) \dashv\vdash_{\mathsf{IPC}} \neg\neg(\neg\neg\phi \to \neg\neg\psi) \quad (6)$$
$$\neg(\phi \wedge \neg\psi) \dashv\vdash_{\mathsf{IPC}} \neg\neg(\phi \to \psi) \quad (7)$$
$$\neg\neg\forall x.\neg\neg\phi \dashv\vdash_{\mathsf{IQC}} \forall x.\neg\neg\phi \quad (8)$$

3 Negative translations

At least as far as modality-free languages are concerned, standard references provide numerous overviews of negative translations, e.g., [52, Ch. 2.3], [49, Ch. 6–7], [14, Ch. 2]. However, in the discussion below we are relying in particular on Ferreira and Oliva [21]. In the modal case, there is some discussion of negative translations for the basic systems $\mathrm{i}_{\Box}\mathsf{K}$ [12, 28].

The most general conditions for a mapping $(\cdot)^t : \mathcal{L}_{\Box} \mapsto \mathcal{L}_{\Box}$ to be considered a translation function are mirroring those proposed by Gaspar [25]:

▶ **Definition 3.** Given a set of axioms $\mathcal{Z} \subseteq \mathcal{L}_{\Box}$ and a translation function $(\cdot)^t : \mathcal{L}_{\Box} \mapsto \mathcal{L}_{\Box}$, say that a translation is $\mathcal{Z}$*-sane* if for any $\phi \in \mathcal{L}_{\Box}$, $\phi \leftrightarrow \phi^t \in \mathrm{c}_{\Box}\mathcal{Z}$. A $\mathcal{Z}$-sane translation is furthermore called $\mathcal{Z}$*-adequate*[5] if for any Γ and ϕ, $\vdash_{\mathsf{Nc}_{\Box}\mathcal{Z}} \Gamma \Rightarrow \phi$ implies $\vdash_{\mathsf{Ni}_{\Box}\mathcal{Z}} \Gamma^t \Rightarrow \phi^t$.

Being both sane and adequate guarantees that $\phi \in \mathrm{c}_{\Box}\mathcal{Z}$ iff $\phi^t \in \mathrm{i}_{\Box}\mathcal{Z}$. Note that $\mathcal{Z}$-sanity is a rather trivial condition, as it transfers upwards: if $\mathrm{i}_{\Box}\mathcal{Z} \subseteq \mathrm{i}_{\Box}\mathcal{Y}$, then $\mathcal{Z}$-sanity ensures $\mathcal{Y}$-sanity; in particular, K-sanity implies $\mathcal{Z}$-sanity for any $\mathcal{Z}$. The very notion does not need to be mention often from now on and can be tacitly assumed: all the translations we are concerned with are K-sane. Adequacy, on the other hand, is more of a challenge; as we are going to see, when some problematic choices are made, even K-adequacy is not guaranteed.

The most direct and throughout solution is the earliest one, proposed by Kolmogorov in 1925: drop $\neg\neg$ in front of *every* subformula. This obviously extends to the modal setting:

$$\bot^{\mathsf{kol}} := \bot \qquad p^{\mathsf{kol}} := \neg\neg p \qquad (\phi \wedge \psi)^{\mathsf{kol}} := \neg\neg(\phi^{\mathsf{kol}} \wedge \psi^{\mathsf{kol}})$$
$$(\phi \vee \psi)^{\mathsf{kol}} := \neg\neg(\phi^{\mathsf{kol}} \vee \psi^{\mathsf{kol}}) \qquad (\phi \to \psi)^{\mathsf{kol}} := \neg\neg(\phi^{\mathsf{kol}} \to \psi^{\mathsf{kol}}) \qquad (\Box\phi)^{\mathsf{kol}} := \neg\neg\Box\phi^{\mathsf{kol}}$$

Of course, the Kolmogorov translation is not a model of parsimony with its liberal use of $\neg\neg$. There are two possible strategies of eliminating redundant occurrences of double negation, which we can baptize the *inner* strategy and the *outer* strategy.

The inner one is the one leading to (several variants of) the *Gödel-Gentzen* translation. If one begins with $s_{\neg\neg}(\cdot)$ allowing $\neg\neg$ to "spread from the inside" then in the inductive definition of translation, $\neg\neg$ is redundant in clauses for $\wedge$ and $\to$ owing to, respectively, Lemma 2(1) and 2(6). There are several choices one can make for $\vee$, thanks to the validity of Lemma 2(4) and 2(5); let us also note that Gödel himself relied on 2(7) to provide an alternative clause for $\to$, but few authors have followed him in this. In the predicate case, it is not necessary to prefix the universal quantifier clause with $\neg\neg$, thanks to Lemma 2(8). The perspective on $\Box$ as a restricted form of $\forall$ would seem to lead to the *naïve Gödel-Gentzen translation*:

$$\bot^{\mathsf{ggn}} := \bot \qquad p^{\mathsf{ggn}} := \neg\neg p \qquad (\phi \wedge \psi)^{\mathsf{ggn}} := \phi^{\mathsf{ggn}} \wedge \psi^{\mathsf{ggn}}$$
$$(\phi \vee \psi)^{\mathsf{ggn}} := \neg\neg(\phi^{\mathsf{ggn}} \vee \psi^{\mathsf{ggn}}) \qquad (\phi \to \psi)^{\mathsf{ggn}} := \phi^{\mathsf{ggn}} \to \psi^{\mathsf{ggn}} \qquad (\Box\phi)^{\mathsf{ggn}} := \Box\phi^{\mathsf{ggn}}.$$

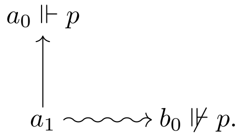
However, this translation is *not even* K*-adequate* (see also [12, p. 231–232], [28, §2.2]). A modal analogue of Lemma 2(8) fails:

[5] Gaspar [25] uses names "characterization" and "soundness", respectively. Similar conditions are also discussed by Ferreira and Oliva [21].

▶ **Example 4.** We have that $\neg\neg\Box p \to \Box p \in \mathrm{c}_\Box\mathsf{K}$, but

$$(\neg\neg\Box p \to \Box p)^{\mathsf{ggn}} = \neg\neg\Box\neg\neg p \to \Box\neg\neg p \notin \mathrm{i}_\Box\mathsf{K}.$$

Consider the following model:



Note it is irrelevant whether points of this model are $\rightsquigarrow$-reflexive or $\rightsquigarrow$-irreflexive. Let us write $\mathfrak{C}^{\circ}_{\mathsf{ggn}}$ for the first variant and $\mathfrak{C}^{\bullet}_{\mathsf{ggn}}$ for the second.

▶ **Theorem 5.** *For any set of axioms $\mathcal{Z}$ which holds either in $\mathfrak{C}^{\circ}_{\mathsf{ggn}}$ or in $\mathfrak{C}^{\bullet}_{\mathsf{ggn}}$, the naïve Gödel-Gentzen translation* ggn *is not $\mathcal{Z}$-adequate.*

Hence, one can consider the *refined* variant of Gödel-Gentzen (see also [12, p. 231–232], [28, Def 2.26]), which only differs in the clause for $\Box$:

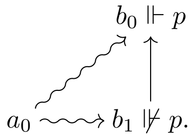
$$\bot^{\mathsf{ggr}} := \bot \qquad p^{\mathsf{ggr}} := \neg\neg p \qquad (\phi \wedge \psi)^{\mathsf{ggr}} := \phi^{\mathsf{ggr}} \wedge \psi^{\mathsf{ggr}}$$
$$(\phi \vee \psi)^{\mathsf{ggr}} := \neg\neg(\phi^{\mathsf{ggr}} \vee \psi^{\mathsf{ggr}}) \qquad (\phi \to \psi)^{\mathsf{ggr}} := \phi^{\mathsf{ggr}} \to \psi^{\mathsf{ggr}} \qquad (\Box\phi)^{\mathsf{ggr}} := \neg\neg\Box\phi^{\mathsf{ggr}}.$$

The outer strategy is the one taken by Glivenko in 1929. In the propositional case, it consists in prefixing *just the entire formula* by $\neg\neg$. In other words, one defines $\phi^{\mathsf{glv}} := \neg\neg\phi$. The double negation connective "penetrates from the outside of" an $\mathcal{L}$-formula thanks to the validity of Lemma 2(2) , 2(4) and 2(6) . But in this case, the strategy requires an adjustment even for IQC: $\neg\neg\forall x.\phi$ is not intuitionistically equivalent to $\neg\neg\forall x.\neg\neg\phi$. It is natural then that counterexamples can be found for $\Box$ too [12, p. 231–232]; although, as we are going to discuss in §6, the Glivenko translation does work for a surprisingly large class of logics where $\Box$ does *not* behave like the universal quantifier and, on the other hand, the are *some* modal logics corresponding to specific universal theories where the Glivenko translation works for *some* formulas [8] (see also §7).

▶ **Example 6.** Clearly, we have that $\Box(\neg\neg p \to p) \in \mathrm{c}_\Box\mathsf{K}$, but

$$\Box(\neg\neg p \to p)^{\mathsf{glv}} = \neg\neg\Box(\neg\neg p \to p) \notin \mathrm{i}_\Box\mathsf{K}.$$

Consider the following model:



Just like before, $\rightsquigarrow$-reflexivity or $\rightsquigarrow$-irreflexivity of points in this models does not play any rôle. Let us write $\mathfrak{C}^{\circ}_{\mathsf{glv}}$ for the first variant and $\mathfrak{C}^{\bullet}_{\mathsf{glv}}$ for the second.

▶ **Theorem 7.** *For any set of axioms $\mathcal{Z}$ which holds either in $\mathfrak{C}^{\circ}_{\mathsf{glv}}$ or in $\mathfrak{C}^{\bullet}_{\mathsf{glv}}$, the Glivenko translation* glv *is not $\mathcal{Z}$-adequate.*

This leads to the Kuroda 1951 translation (refined Glivenko), which we can define as $\phi^{\mathsf{kur}} := \neg\neg\phi_{\mathsf{kur}}$, using an auxiliary translation $(\cdot)_{\mathsf{kur}}$, which is defined as $(\Box\phi)_{\mathsf{kur}} := \Box\neg\neg\phi_{\mathsf{kur}}$ and identity in other inductive clauses, i.e.,

$$\bot_{\mathsf{kur}} := \bot \qquad p_{\mathsf{kur}} := p \qquad (\phi \wedge \psi)_{\mathsf{kur}} := \phi_{\mathsf{kur}} \wedge \psi_{\mathsf{kur}}$$
$$(\phi \vee \psi)_{\mathsf{kur}} := \phi_{\mathsf{kur}} \vee \psi_{\mathsf{kur}} \qquad (\phi \to \psi)_{\mathsf{kur}} := \phi_{\mathsf{kur}} \to \psi_{\mathsf{kur}} \qquad (\Box\phi)_{\mathsf{kur}} := \Box\neg\neg\phi_{\mathsf{kur}}.$$

3.1 Monotone modular and regular translations

Negative translations of interest to us form a subclass of *modular* ones as defined by Ferreira and Oliva [21]. We can define a somewhat narrower umbrella notion: *monotone modular* *translations*. Such a translation is generated by a function

$$\mathsf{cont}t : ((\{\Box,\wedge,\vee,\rightarrow\}\times\{i,o\})\cup\{\Sigma,\vdash\})\mapsto\{0,1\},$$

with the intuition that 0 and 1, respectively, stand for the number of occurrences of $\neg\neg$ and i and o, respectively, abbreviate *inside/outside*. We infix the first argument of $\mathsf{cont}t$ as subscript. Similarly to Ferreira and Oliva [21], $\mathsf{cont}t_{\vdash}$ stands for $\neg\neg$ used (or not) in front of the entire formula, cf. the distinction between $(\cdot)_{\mathsf{kur}}$ and $(\cdot)^{\mathsf{kur}}$ above.

Define now $\phi^t := {}^{\mathsf{cont}t_{\vdash}\cdot\neg\neg}\phi_t$, where

$$\begin{array}{llll}
\bot_t & := \bot & (\phi\wedge\psi)_t & := {}^{\mathsf{cont}t_{\wedge}o\cdot\neg\neg}({}^{\mathsf{cont}t_{\wedge}i\cdot\neg\neg}\phi_t \wedge {}^{\mathsf{cont}t_{\wedge}i\cdot\neg\neg}\psi_t)\\
p_t & := {}^{\mathsf{cont}t_{\Sigma}\cdot\neg\neg}p & (\phi\vee\psi)_t & := {}^{\mathsf{cont}t_{\vee}o\cdot\neg\neg}({}^{\mathsf{cont}t_{\vee}i\cdot\neg\neg}\phi_t \vee {}^{\mathsf{cont}t_{\vee}i\cdot\neg\neg}\psi_t)\\
(\Box\phi)_t & := {}^{\mathsf{cont}t_{\Box}o\cdot\neg\neg}\Box({}^{\mathsf{cont}t_{\Box}i\cdot\neg\neg}\phi_t) & (\phi\rightarrow\psi)_t & := {}^{\mathsf{cont}t_{\rightarrow}o\cdot\neg\neg}({}^{\mathsf{cont}t_{\rightarrow}i\cdot\neg\neg}\phi_t \rightarrow {}^{\mathsf{cont}t_{\rightarrow}i\cdot\neg\neg}\psi_t).
\end{array}$$

▶ **Fact 8.** *Any monotone modular translation is* K*-sane, hence* $\mathcal{Z}$*-sane for any* $\mathcal{Z}\subseteq\mathcal{L}_{\Box}$*.*

The Glivenko translation is defined by $\mathsf{conglv}_{\vdash} := 1$ and 0 for all other arguments. The Kuroda translation is defined by $\mathsf{conkur}_{\vdash} := 1$, $\mathsf{conkur}_{\Box}i := 1$ and 0 elsewhere. The naïve Gödel-Gentzen translation is defined by $\mathsf{conggn}_{\Sigma} := 1$, $\mathsf{conggn}_{\vee}o := 1$ and 0 elsewhere. The refined Gödel-Gentzen translation is defined by $\mathsf{conggr}_{\Sigma} := 1$, $\mathsf{conggr}_{\vee}o := 1$, $\mathsf{conggr}_{\Box}o := 1$ and 0 elsewhere. The Kolmogorov translation is defined by $\mathsf{conggr}_{\Sigma} := 1$, $\mathsf{conggr}_{*}o$ for $*\in\{\Box,\wedge,\vee,\rightarrow\}$ and 0 elsewhere. There is a natural *weak refinement ordering* on such translations: $t\leq t'$ whenever $\mathsf{cont}t$ is pointwise below $\mathsf{cont}t'$ (obviously, $0\leq 1$). Thus, $\mathsf{ggn}\leq\mathsf{ggr}\leq\mathsf{kol}$ and $\mathsf{glv}\leq\mathsf{kur}$. Any monotone modular translation weakly refining either ggr or kur is called *regular*. As the question of adequacy of regular translations will be of central importance below, let us introduce a name for it.

▶ **Definition 9.** A logic $\mathsf{i}_{\Box}\mathcal{Z}$ is $\neg\neg$*-complete* if some/any regular t is adequate for it.

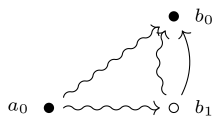
▶ **Theorem 10.** *Any regular translation* t *is* K*-adequate, i.e.,* $\mathsf{i}_{\Box}\mathsf{K}$ *is* $\neg\neg$*-complete. Furthermore, for every* $\phi\in\mathcal{L}_{\Box}$ *we have that* $\vdash_{\mathsf{K}}\phi^t\leftrightarrow\neg\neg\phi^t$ *and for any other regular translation* t'*, we have that* $\vdash_{\mathsf{K}}\phi^t\leftrightarrow\phi^{t'}$*.*

As said above, $\neg\neg$-completeness of the minimal system $\mathsf{i}_{\Box}\mathsf{K}$ has already been noted in the literature [12, 28] But not all logics are $\neg\neg$-complete.

4 Failure of $\neg\neg$-completeness

This section presents counterexamples illustrating that there are logics for which regular translations are not adequate. Apart from counterexamples in the previous section, this is the only place where we need to use Kripke semantics.

▶ **Example 11.** Consider the following frame $\mathfrak{C}_{\mathsf{den}}$:



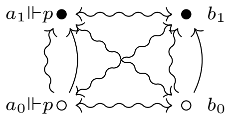
. We have that $\mathfrak{C}_{\mathsf{den}}\Vdash\mathsf{i}_{\Box}\mathsf{C4}$, where $\mathsf{C4} := \Box\Box p\rightarrow\Box p$. But $a_0\nVdash\neg\neg\Box\neg\neg\Box\neg\neg p\rightarrow\neg\neg\Box\neg\neg p$. To see this, just set $V(p) := \emptyset$ and notice this yields $\overline{V}(\neg\neg p)=\emptyset$, $\overline{V}(\Box\neg\neg p)=\{b_0\}=b_0\!\uparrow$ and $\overline{V}(\neg\neg\Box\neg\neg p)=\{b_0,b_1\}=b_1\!\uparrow$, whereas $\overline{V}(\Box\neg\neg\Box\neg\neg p)$ is the entire carrier of $\mathfrak{C}_{\mathsf{den}}$ and hence so is $\overline{V}(\neg\neg\Box\neg\neg\Box\neg\neg p)$.

This lifts to a theorem about an interval of logics:

▶ **Theorem 12.** *No extension of* $\mathrm{i}_\Box\mathsf{K}$ *contained between* $\mathrm{i}_\Box\mathsf{C4}$ *and the logic of* $\mathfrak{C}_{\mathsf{den}}$ *is* $\neg\neg$*-complete.*

Is this an isolated example? As it turns out, $\neg\neg$-completeness can fail for other rather simple *modal reduction principles* [57, §4.5] as well.

▶ **Example 13.** Consider $\mathsf{Minus4} := \Box\Box p \to \Box\Box\Box p$ and the following frame $\mathfrak{C}_{\mathsf{tr2}}$ with $V(p) = a_0\!\uparrow$:



. We can easily show that $\mathfrak{C}_{\mathsf{tr2}} \Vdash \mathsf{Minus4}$ by noting that every point in $\mathfrak{C}_{\mathsf{tr2}}$ can reach every other one (including itself) in at most two $\rightsquigarrow$-steps. On the other hand,

$$\mathfrak{C}_{\mathsf{tr2}}, V, a_0 \nvDash \neg\neg\Box\neg\neg\Box\neg\neg p \to \neg\neg\Box\neg\neg\Box\neg\neg\Box\neg\neg p.$$

This is verified as follows: $V(p) = \overline{V}(\neg\neg p) = \overline{V}(\neg\neg\Box\neg\neg\Box\neg\neg p) = a_0\!\uparrow$, $\overline{V}(\Box\neg\neg p) = \overline{V}(\Box\neg\neg\Box\neg\neg\Box\neg\neg p) = \{b_1\}$, $\overline{V}(\neg\neg\Box\neg\neg p) = \overline{V}(\neg\neg\Box\neg\neg\Box\neg\neg\Box\neg\neg p) = b_0\!\uparrow$ and $\overline{V}(\Box\neg\neg\Box\neg\neg p) = \{a_1\}$.

▶ **Theorem 14.** *No logic contained between* $\mathrm{i}_\Box\mathsf{Minus4}$ *and the logic of* $\mathfrak{C}_{\mathsf{tr2}}$ *is* $\neg\neg$*-complete.*

4.1 Aside: with great power comes great inadequacy

While in this paper we focus almost entirely on the simplest possible case of $\Box$-logics, inadequacy can be a more dramatic phenomenon for very natural systems with more connectives, e.g., residuated binary ones of BI: the logic of *bunched implications* [42, 46]. On the one hand, Galmiche et al. [24] have shown that BI is decidable; this result allows strengthening and generalizations [22]. On the other hand, the undecidability of its classical extension BBI follows already from results of Kurucz et al. [2, 33] (shown using von Neumann's *n-frames* originating in projective geometry) and has been recently rediscovered and extended to logics determined by concrete heap models using more computational techniques [13, 35].

Hence, no *recursive* translation from BBI to BI (much less a modular negative one) can be adequate; otherwise, one could use such a translation to define a decision procedure for BBI.

5 Syntactic criteria of $\neg\neg$-completeness

Given that $\neg\neg$-completeness can fail so dramatically for very natural axioms, it is natural to ask how one can obtain general *positive* results. First, let us analyze where the problem comes from. By definition, a logic fails to be $\neg\neg$-complete iff there exists $\phi \in \mathrm{c}_\Box\mathcal{Z}$ s.t $\phi^t \notin \mathrm{i}_\Box\mathcal{Z}$. It would seem that nothing forces this ϕ to be itself an axiom, i.e., a member of $\mathcal{Z}$, but this is precisely what we see in the counterexamples in the preceding section. The following theorem clarifies it is not a coincidence—and provides us with an useful criterion of $\neg\neg$-completeness.

▶ **Theorem 15** (Regular Adequacy)**.**

- *A logic* $\mathrm{i}_\Box\{\zeta\}$ *is* $\neg\neg$*-complete iff* $\zeta^t \in \mathrm{i}_\Box\{\zeta\}$ *for some/any regular translation* t*.*
- *A logic* $\mathrm{i}_\Box\mathcal{Z}$ *is* $\neg\neg$*-complete whenever for any* $\zeta \in \mathcal{Z}$*,* $\zeta^t \in \mathrm{i}_\Box\mathcal{Z}$ *for some/any regular translation* t*.*

Proof. See Appendix A or the associated Coq formalization. ◀

Thus, whenever a given finite set of axioms $\mathcal{Z}$ axiomatizes a logic with a known decision algorithm, one can use the decision procedure for $\mathrm{i}_\Box\mathcal{Z}$ to check its $\neg\neg$-completeness: i.e., by checking if the Kolmogorov translation (or any other regular translation, e.g., Kuroda for simplicity) of each axiom from $\mathcal{Z}$ is a member of $\mathrm{i}_\Box\mathcal{Z}$. Of course, one needs to keep in mind that the problem whether a given formula axiomatizes a decidable logic is undecidable itself (see, e.g., [14, Ch. 17], [57, §3] for references). Nevertheless, as we are going to discuss now, Theorem 15 does yield general positive results on $\neg\neg$-completeness of logics with axioms of a specific syntactic shape.

▶ **Definition 16.** We call $\beta \in \mathcal{L}_\Box$ a *$\neg\neg$-pre-envelope (in $\mathrm{i}_\Box\mathcal{Z}$)* if $\neg\neg s_{\neg\neg}(\beta) \vdash_{\mathrm{i}_\Box\mathcal{Z}} \beta^{\mathsf{kol}}$; recall that $s_{\neg\neg}(\beta)$ is a substitution replaces all p occurring in β by their double negations. When not stated specifically, we will take $\mathrm{i}_\Box\mathcal{Z}$ to be $\mathrm{i}_\Box\mathsf{K}$. Analogously, we call $\beta \in \mathcal{L}_\Box$ a *$\neg\neg$-post-envelope (in $\mathrm{i}_\Box\mathcal{Z}$)* if $\beta^{\mathsf{kol}} \vdash_{\mathrm{i}_\Box\mathcal{Z}} \neg\neg s_{\neg\neg}(\beta)$. A *$\neg\neg$-envelope* is a formula which is both pre- and post-envelope, i.e., a formula $\beta \in \mathcal{L}_\Box$ s.t. $\neg\neg s_{\neg\neg}(\beta) \dashv\vdash_{\mathrm{i}_\Box\mathcal{Z}} \beta^{\mathsf{kol}}$. An *enveloped implication* is of the form $\beta \to \gamma$ where β is a post-envelope and γ is a pre-envelope.

Clearly, a $\neg\neg$-envelope can be consider a special (or degenerate) case of an enveloped implication. To illustrate these notions, recall that a *shallow formula* is one with no nesting of $\Box$, i.e., one where every $p \in \Sigma$ is within the scope of at most one $\Box$, whereas a *box-free formula* is one without any occurrences of $\Box$ at all. Now we have:

▶ **Lemma 17** (Envelope Criteria).

- *A box-free formula is a $\neg\neg$-envelope*
- *A shallow formula with no disjunction under box is a $\neg\neg$-envelope*
- *An implication-free formula is a pre-envelope*
- *A negation of a pre-envelope is a post-envelope*

▶ **Theorem 18** (Enveloped Implications). *Any logic axiomatized by enveloped implications is $\neg\neg$-complete.*

Proof. See Appendix B or the associated formalization for details. ◀

Theorem 18 jointly with Lemma 17 yields $\neg\neg$-completeness of numerous logics, including, for example, many of those given in Table 2 of Litak [36] or in Theorem 10 of Sotirov [50]. Here are just some examples:

▶ **Corollary 19.** *$\neg\neg$-completeness holds for any logic axiomatized over $\mathrm{i}_\Box\mathsf{K}$ by any combination of the following formulas:*

R	$p \to \Box p$,	CB	$\Box p \to (q \to p) \vee q$,	$\Box$em	$\Box p \vee \Box\neg p$,
4	$\Box p \to \Box\Box p$,	NV	$\neg\Box\bot$,	em$\Box$	$\Box p \vee \neg\Box p$,
T	$\Box p \to p$,	NNV	$\neg\neg\Box\bot$,	T$\neg\neg$	$\Box p \to \neg\neg p$,
antK	$(\Box p \to \Box q) \to \Box(p \to q)$,	MinR	$p \to \Box\Box p$,	T$\neg$	$\Box\neg p \to \neg p$,
$\Box$Lin	$\Box(p \to q) \vee \Box(q \to p)$,	Lin$\Box$	$(\Box p \to q) \vee (\Box q \to p)$,	$\neg\Box\neg$em	$\neg\Box p \vee \neg\Box\neg p$,

or any superintuitionistic axiom, i.e., a formula in modality-free $\mathcal{L}$.

As an example of a standard logic obtained by a combination of axioms given in Corollary 19, consider $\mathrm{i}_\Box\mathsf{S4} = \mathrm{i}_\Box\mathsf{4} + \mathsf{T}$.

Counterexamples in §4 were easily seen to produce entire intervals of $\neg\neg$-incomplete logics (Theorems 12 and 14). In contrast, Corollary 19 does not automatically lead to non-trivial examples of entire intervals of $\neg\neg$-*complete* logics, especially logics whose *all* extensions are $\neg\neg$-complete. Corollary 19 mentions, e.g., $\mathrm{i}_\Box\mathsf{4}$ and Example 11 together with Theorem 12 provide an example of an extension of $\mathrm{i}_\Box\mathsf{4}$ for which $\neg\neg$-completeness fails: the logic of $\mathfrak{C}_{\mathsf{den}}$.

However, Corollary 19 also mentions $\mathsf{i}_\Box\mathsf{R}$—and for this system, we can do much better. As it turns out, this is related to another subject: for some classes of logics, irregular translations can be adequate.

6 Strength makes irregularity adequate

Clearly, the failure of the Glivenko translation amounts to the failure of the "outer strategy" described in §3 to penetrate through $\Box$. The axiom $\forall x.\neg\neg\phi \to \neg\neg\forall x.\phi$ which ensures this strategy succeeds in the predicate case is thus called the *Double Negation Shift* [52, Ch. 2] and in some references also *Kuroda principle* (*scheme, axiom*)—see, e.g., [20] for an example of both names in use. It is natural to use similar name(s) for its modal analogue. In other words, the (*modal*) *Double Negation Shift* (or the *modal Kuroda axiom)* is

DNS $\quad \Box\neg\neg p \to \neg\neg\Box p.$

The corresponding logic is naturally denoted as $\mathsf{i}_\Box\mathsf{DNS}$. Clearly, we have that for any $\mathsf{i}_\Box\mathcal{Z}$ extending $\mathsf{i}_\Box\mathsf{DNS}$ and any $\phi \in \mathcal{L}_\Box$, $\neg\neg\Box\neg\neg\phi \leftrightarrow \neg\neg\Box\phi \in \mathsf{i}_\Box\mathcal{Z}$. In fact, it is straightforward to see that these two axiom schemes are equivalent:

▶ **Lemma 20.** DNS *and* $\neg\neg\Box\neg\neg p \leftrightarrow \neg\neg\Box p$ *axiomatize the same logic.*

We have an analogue of Exercise 2.3.3 in Troelstra and van Dalen [52]:

▶ **Theorem 21.** *In any extension of* $\mathsf{i}_\Box\mathsf{DNS}$*, the Glivenko translation becomes equivalent to the regular ones, i.e., for any regular* t *and every* ϕ*, we have that:* $\vdash_{\mathsf{i}_\Box\mathsf{DNS}} \phi^t \leftrightarrow \phi^{\mathsf{glv}}$.

Hence, we have $\neg\neg$-completeness of *all logics* in which the Kuroda axiom is derivable:

▶ **Theorem 22.** *Any* t *refining* glv *is adequate for any extension of* $\mathsf{i}_\Box\mathsf{DNS}$.

Proof sketch. Theorem 21 yields that for any β, we have that $\neg\neg\beta \dashv\vdash_{\mathsf{i}_\Box\mathsf{DNS}} \beta^t$. Now it is enough to notice that $\beta \to \neg\neg\beta$ holds in any extension of $\mathsf{i}_\Box\mathsf{K}$. ◀

Are extensions of $\mathsf{i}_\Box\mathsf{DNS}$ of interest? Consider $\mathsf{i}_\Box\mathsf{R} = \mathsf{i}_\Box\mathsf{K} + p \to \Box p$, which can be seen as the logic of *applicative functors*, also known as *idioms* [37]. Our presently used name R follows the usage in Litak [36], which in turn uses the same name as Fairtlough and Mendler [19]; another possible justification for this name would come from the "return" law of monads. An alternative name would be S, coming from, on the one hand, the *strong Löb axiom*

SL $\quad (\Box p \to p) \to p,$

and on the other hand from categorical "strength" of $\Box$ thought of as a functor, whose trace on the level of type/object inhabitation is

Str $\quad p \wedge \Box q \to \Box(p \wedge q).$

It is easy to see that R and Str are interderivable over $\mathsf{i}_\Box\mathsf{K}$.[6] Regarding the strong Löb axiom, the derivation of S in $\mathsf{i}_\Box\mathsf{SL}$ can be found, e.g., in Milius and Litak [38, §3] (in a categorical disguise). It is a simplified variant of the deduction of 4 from the ordinary Löb axiom found independently by de Jongh, Kripke and Sambin in mid-1970's, cf. [10, Th 18]. SL is an axiom whose importance has been first noticed in provability logics HA^* and PA^* [29, 55]. Its recent popularity deriving from Nakano [40, 41] comes from its rôle in

[6] We decided against the use of S to avoid confusion with standard modal names S4 and S5, which come from unfortunate historical coincidences.

guarding (co-)recursion and (co-)induction.[7] Another important class of extensions of $\mathsf{i}_\Box\mathsf{R}$ is provided by extensions of $\mathsf{PLL} := \mathsf{i}_\Box\mathsf{R} + \mathsf{C4}$ (Propositional Lax Logic [19]): the logic of the *Grothendieck topology* [26], but also *access control* and the Curry-Howard counterpart of Moggi's *computational metalanguage* (see [36] for references). Recall that above $\mathsf{i}_\Box\mathsf{K}$, $\mathsf{C4}$ was one of our flagship examples of an axiom failing $\neg\neg$-completeness. Finally, Artemov and Protopopescu [4] use the BHK interpretation to justify the epistemic importance of certain extension of $\mathsf{i}_\Box\mathsf{R}$ and PLL. In short, despite the non-classical character of these modalities, we are dealing with a large class of important extensions of $\mathsf{i}_\Box\mathsf{K}$; as mentioned in the introduction, this puts in doubt Simpson's Requirement 3 [47, Ch 3.2]. And we have:

▶ **Theorem 23.** DNS *is a theorem of* $\mathsf{i}_\Box\mathsf{R}$.

Proof. See Appendix C or the associated Coq formalization. ◀

▶ **Corollary 24.** *Any t refining* glv *is adequate for any extension of* $\mathsf{i}_\Box\mathsf{R}$*; in particular, each such logic is $\neg\neg$-complete.*

▶ **Remark 25.** Extensions of $\mathsf{i}_\Box\mathsf{R}$ have a rather trivial "double negation core": $\mathsf{c}_\Box\mathsf{K}$ extended with the "strength" or "return" axiom R is equipollent with CPC enriched with an additional propositional constant (corresponding to $\Box\bot$). There are only three consistent (and coNP-complete) logics of this kind, whereas the lattice of extensions of (PSPACE-hard) $\mathsf{i}_\Box\mathsf{R}$ is uncountable and richer than the lattice of extensions of IPC, which can be identified with $\mathsf{PLL} + \Box p \leftrightarrow p$; thus, it also includes undecidable logics. Hence, in many ways the gap between $\mathsf{i}_\Box\mathsf{S}$ and $\mathsf{c}_\Box\mathsf{S}$ is more dramatic than that between $\mathsf{i}_\Box\mathsf{S4}$ and $\mathsf{c}_\Box\mathsf{S4}$. One can see the fact that the "double negation core" of any extension of $\mathsf{i}_\Box\mathsf{R}$ collapses so much information as a deeper reason behind Corollary 24.

7 Conclusions and future work

Some of the points below are going to be discussed in detail in the journal version of this paper or in a companion one, others are more speculative. While we consciously restricted our attention to syntactic criteria and syntactic proofs in this paper, it is also possible to prove positive general results for adequacy and $\neg\neg$-completeness in terms of stability under $\uparrow$*-cofinal* and $\uparrow$*-generated subframes*; indeed, readers familiar with modal logic probably noticed that counterexamples in §4 fail to be *subframe logics* (cf. e.g, [59, §1.8]). Does this approach lead to a more general characterization of adequacy than Theorem 18? On another note, one of reasons why modular translations of Ferreira and Oliva [21] are defined in a more general way than our monotone modular translations in §3.1 is that they cover also the so-called Krivine(-Streicher-Reus) translation. This translation dualizes connectives, in particular uses $\forall$ and $\exists$ to translate each other. It would seem natural to study such translations, but this appears sensible only in presence of $\Diamond$ suitably related to $\Box$. In this connection, let us recall that Bezhanishvili [8] studies the scope of Glivenko-type theorems for specific formulas in extensions of Prior's MIPC, i.e., logics equipped with $\Box$ and $\Diamond$ which behave as close as possible to genuine $\forall$ and $\exists$ for concrete theories in IQC. Furthermore, it would be of obvious interest to study still more complicated supplies of connectives and

[7] It is used to ensure productivity in (co-)programming and on the metalevel—in semantic reasoning about programs involving higher-order store or a combination of impredicative quantification with recursive types. There are too many recent references to quote here, so let us just point out that Milius and Litak [38] provide a general framework for various categorical models in the literature, ranging from ultrametric spaces to the *topos of trees* of Birkedal and coauthors [9]. For more references, see also [36].

axioms than those involving $\Box$ and $\Diamond$, but as discussed in §4.1 using BI as an example, things can go very wrong for very natural logics and the scope of any general positive results will be by nature limited, whereas a suitable proof-theoretic setup and associated proof-assistant formalization would be more complicated. Speaking of substructural connectives, it would seem natural to merge the present line of research with syntactic research on substructural negative translations by Ono and coauthors [20, 43]. Still a different direction would be to pick up the theme already signalled in Footnote 2: $\neg\neg$ can and has been studied as a modality in its own right [11, 17], especially over negation-free and bottom-free intutionistic sytax. Algebraically, one can see $\neg\neg$ as a *nucleus* and categorically—as (inhabitation/object level trace of) the *continuation monad*. The corresponding generalization of our study of modal negative translations would tie with the work of Aczel [1] and Escardó and Oliva [18]. Replacement of the intuitionistic logic with the minimal logic would be of interest given the significance of the latter in terms of control operators [3]. However, one cannot expect arbitrary modal axioms to enjoy some computational significance. Pfenning and Davies [45] stress the importance of adequate introduction and elimination rules in this context, Martin-Löf style; contrast this with our discussion in §2.2 of the unavailability of an apparatus producing such rules for arbitrary axioms without any syntactic restrictions on their shape. At the very least, however, we hope that the present development provides good limitative results and a general framework for investigating the relationship between constructive and classical variants of concrete axioms/inhabitation laws. On a more general note, we believe that our study illustrates the availability of purely constructive and syntactic methods in modal logic understood as "die Klassentheorie" [57]. We do not claim that such methods can or should *replace* algebraic, topological and model-theoretic ones. But it is good to remember that they can be employed in the service of what Wolter and Zakharyaschev call the *global view* and *intuitionistic* modal logics seem a particularly natural target.

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Appendix

As stated in the text, proofs are available in the associated Coq formalization, see Footnote 1 for details and download information. However, for convenience of referees we provide here several more humanly digestible variants.

A Proof of Theorem 15

Fix $t = \mathsf{ggr}$. We need to show that whenever $\vdash_{\mathsf{Nc}_\Box\mathcal{Z}} \Gamma \Rightarrow \phi$, $\vdash_{\mathsf{Ni}_\Box\mathcal{Z}} \Gamma^{\mathsf{ggr}} \Rightarrow \phi^{\mathsf{ggr}}$. The proof proceeds by induction on the derivation of $\vdash_{\mathsf{Nc}_\Box\mathcal{Z}} \Gamma \Rightarrow \phi$. The assumption of the theorem guarantees the AxSb case. The cases of IPC rules are straightforward and known, with the case of $(\vee_E)$ requiring, as usual, somewhat more bookkeeping. Let us show the modal case, i.e., $\Box_{\mathsf{K}}$. Our goal is to derive $\vdash_{\mathsf{Ni}_\Box\mathcal{Z}} \Gamma^{\mathsf{ggr}} \Rightarrow \neg\neg\Box\psi^{\mathsf{ggr}}$ under the assumption that

$$\vdash_{\mathsf{Nc}_\Box\mathcal{Z}} \Gamma \Rightarrow \Box\phi_1, \ldots, \vdash_{\mathsf{Nc}_\Box\mathcal{Z}} \Gamma \Rightarrow \Box\phi_n \text{ and } \vdash_{\mathsf{Nc}_\Box\mathcal{Z}} \phi_1, \ldots, \phi_n \Rightarrow \psi$$

and hence, by IH,

$$\vdash_{\mathsf{Ni}_\Box\mathcal{Z}} \Gamma^{\mathsf{ggr}} \Rightarrow \neg\neg\Box\phi_1^{\mathsf{ggr}}, \ldots, \vdash_{\mathsf{Ni}_\Box\mathcal{Z}} \Gamma^{\mathsf{ggr}} \Rightarrow \neg\neg\Box\phi_n^{\mathsf{ggr}} \text{ and } \vdash_{\mathsf{Ni}_\Box\mathcal{Z}} \phi_1^{\mathsf{ggr}}, \ldots, \phi_n^{\mathsf{ggr}} \Rightarrow \psi^{\mathsf{ggr}}.$$

But now it is enough to observe that the following rule is derivable:

$$\neg\neg\Box_{\mathsf{K}}\ \frac{\vdash_{\mathsf{Ni}_\Box\mathcal{Z}} \Gamma \Rightarrow \neg\neg\Box\phi_1 \qquad \ldots \vdash_{\mathsf{Ni}_\Box\mathcal{Z}} \Gamma \Rightarrow \neg\neg\Box\phi_n \qquad \vdash_{\mathsf{Ni}_\Box\mathcal{Z}} \phi_1, \ldots, \phi_n \Rightarrow \psi}{\vdash_{\mathsf{Ni}_\Box\mathcal{Z}} \Gamma \Rightarrow \neg\neg\Box\psi}.$$

B Proof of Theorem 18

By Theorem 15, it is enough to show that $\zeta^{\mathsf{kol}} \in \mathsf{i}_\Box\mathcal{Z}$ for any $\zeta \in \mathcal{Z}$, where $\zeta = \beta \to \gamma$, β is a post-envelope and γ is a pre-envelope.

$$\to_E\ \dfrac{\neg\neg\to_1\ \dfrac{\to_I\ \dfrac{\vdash_{\mathsf{Ni}_\Box\mathcal{Z}} \neg\neg s_{\neg\neg}(\beta) \to \neg\neg s_{\neg\neg}(\gamma), \beta^{\mathsf{kol}} \Rightarrow \gamma^{\mathsf{kol}}}{\vdash_{\mathsf{Ni}_\Box\mathcal{Z}} \Rightarrow (\neg\neg s_{\neg\neg}(\beta) \to \neg\neg s_{\neg\neg}(\gamma)) \to \beta^{\mathsf{kol}} \to \gamma^{\mathsf{kol}}}}{\vdash_{\mathsf{Ni}_\Box\mathcal{Z}} \Rightarrow \neg\neg(\neg\neg s_{\neg\neg}(\beta) \to \neg\neg s_{\neg\neg}(\gamma)) \to \neg\neg(\beta^{\mathsf{kol}} \to \gamma^{\mathsf{kol}})} \qquad \dfrac{\dfrac{\vdash_{\mathsf{Ni}_\Box\mathcal{Z}} \Rightarrow \beta \to \gamma}{\vdash_{\mathsf{Ni}_\Box\mathcal{Z}} \Rightarrow s_{\neg\neg}(\beta) \to s_{\neg\neg}(\gamma)}\ \text{AxSb}}{\vdash_{\mathsf{Ni}_\Box\mathcal{Z}} \Rightarrow \neg\neg(\neg\neg s_{\neg\neg}(\beta) \to \neg\neg s_{\neg\neg}(\gamma))}\ \neg\neg\to_2}{\vdash_{\mathsf{Ni}_\Box\mathcal{Z}} \Rightarrow \neg\neg(\beta^{\mathsf{kol}} \to \gamma^{\mathsf{kol}})}$$

Thus, we just need to show that $\vdash_{\mathsf{Ni}_\Box\mathcal{Z}} \neg\neg s_{\neg\neg}(\beta) \to \neg\neg s_{\neg\neg}(\gamma), \beta^{\mathsf{kol}} \Rightarrow \gamma^{\mathsf{kol}}$. But this follows using the definitions of pre- and post-envelopes.

C Proof of Theorem 23

First, observe that

$$\to_I\ \dfrac{\to_E\ \dfrac{\vdash_{\mathsf{Ni}_\Box\mathsf{R}} \Box\neg\neg\phi, \neg\Box\phi \Rightarrow \Box\phi \qquad \vdash_{\mathsf{Ni}_\Box\mathsf{R}} \Box\neg\neg\phi, \neg\Box\phi \Rightarrow \neg\Box\phi}{\vdash_{\mathsf{Ni}_\Box\mathsf{R}} \Box\neg\neg\phi, \neg\Box\phi \Rightarrow \bot}}{\vdash_{\mathsf{Ni}_\Box\mathsf{R}} \Box\neg\neg\phi \Rightarrow \neg\neg\Box\phi.}$$

Hence, we just need to derive $\vdash_{\mathsf{Ni}_\Box\mathsf{R}} \Box\neg\neg\phi, \neg\Box\phi \Rightarrow \Box\phi$. For this, it is enough to show

$$\vdash_{\mathsf{Ni}_\Box\mathsf{R}} \Box\neg\neg\phi, \neg\Box\phi \Rightarrow \Box\bot.$$

For this in turn, it would be enough to use

$$\Box_{\mathsf{K}}\ \frac{\vdash_{\mathsf{Ni}_\Box\mathsf{R}} \Box\neg\neg\phi, \neg\Box\phi \Rightarrow \Box\neg\phi \qquad \vdash_{\mathsf{Ni}_\Box\mathsf{R}} \Box\neg\neg\phi, \neg\Box\phi \Rightarrow \Box\neg\neg\phi \qquad \vdash_{\mathsf{Ni}_\Box\mathsf{R}} \neg\neg\phi, \neg\phi \Rightarrow \bot}{\vdash_{\mathsf{Ni}_\Box\mathsf{R}} \Box\neg\neg\phi, \neg\Box\phi \Rightarrow \Box\bot.}$$

Only the first premise is interesting, and this is the only part of the proof where the R axiom is actually used:

$$\rightarrow_E \dfrac{\vdash_{\mathsf{Ni}_\Box\mathsf{R}} \neg\Box\phi \Rightarrow \neg\phi \qquad \dfrac{}{\vdash_{\mathsf{Ni}_\Box\mathsf{R}} \neg\Box\phi \Rightarrow \neg\phi \rightarrow \Box\neg\phi}\ \textsc{AxSb}}{\vdash_{\mathsf{Ni}_\Box\mathsf{R}} \neg\Box\phi \Rightarrow \Box\neg\phi.}$$

The derivation of $\vdash_{\mathsf{Ni}_\Box\mathsf{R}} \neg\Box\phi \Rightarrow \neg\phi$ can be now left as an exercise. The deduction used in the above proof should look familiar to readers acquainted with CPS reasoning and the use of control operators.